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The Averaged Motion of a Charged  
Particle in a Dipole Field

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## INTRODUCTION AND NOTATION

The average motion of a charged particle trapped in the geomagnetic field can usually be represented with adequate accuracy by assuming the conservation of the first two adiabatic invariants [e.g. Northrop, 1963]

$$\mu = p_{\perp}^2 / 2 m B \quad (1)$$

$$J = \oint p_{\parallel} ds \quad (2)$$

where integration is along a field line between mirror points. In addition, specification of the averaged motion also involves the parameters defining the magnetic field line to which the particle is attached. This role is usually served by the Euler potentials  $(\alpha, \beta)$  [e.g. Stern, 1970]; given  $(J, \mu)$  of the particle and  $(\alpha, \beta)$  of the field line to which it is attached at some initial time  $t_0$ , the "drift surface" described by its averaged motion is completely determined.

In the absence of electric fields and time dependence the energy is conserved and the drift surface may be described by properties of the trajectory not involving the energy, such as the mirroring field  $B_m$  and the quantity  $I' = J/v$ . In this case  $B_m$  and the initial values of  $(\alpha, \beta)$  are sufficient to define the drift surface, since  $I'$  may be derived from them.

If in addition the magnetic field has the dipole configuration it is often convenient to use instead of  $B_m$  the sine  $y$  of the equatorial pitch angle. An appropriate set of Euler potentials is then, in polar coordinates

$$\alpha = (a^2 g_1^0 / r) \sin^2 \theta \quad (3)$$

$$\beta = a \varphi \quad (4)$$

where  $a$  is the earth's radius and  $g_1^0 \approx -0.31$  gauss is the dipole harmonic coefficient. In many applications it is convenient to introduce a related quantity

$$L = a g_1^0 / \alpha \quad (5)$$

equal to the equatorial distance, in earth radii, of the field line with the given value of  $\alpha$  (one could use  $L$  as an Euler potential, but the form of  $\beta$  then is not as simple). With the above definitions one can define the dimensionless "longitudinal integral"

$$I(y) = \frac{1}{a L} \int_{B'_m}^{B_m} [1 - (B/B_m)]^{\frac{1}{2}} ds \quad (6)$$

evaluated between mirror points and proportional to  $I'$ . Here the initial values of  $\alpha$  and of  $y$  are all that is needed to define the drift surface, since  $I$  depends on  $y$  and  $\beta$  is absent due to axial symmetry. Approximations of the function  $I(y)$  have been derived by Schulz [1971] and by others.

In the geomagnetic field this approach is often used at high energies, where the effects of electric fields can be neglected. However, at energies of the order of one kilovolt this method no longer works, since <sup>kinetic</sup> the energy may change appreciably: in this range  $\mu$  and  $J$ , rather than  $B_m$  and  $I'$  (or  $I$ ) are to be regarded as conserved quantities. The purpose of this work is to provide the means for treating this more general motion, assuming a dipole field.

# THE AVERAGED HAMILTONIAN

The chief tool in this treatment is the averaged Hamiltonian  $K(J, \mu, \alpha, \beta)$  the properties of which have been developed by Northrop and Teller [1960, eqs. 30; see also Northrop, 1963, eqs. 3.68]. Assuming the adiabatic invariants  $J$  and  $\mu$  are conserved and denoting by  $W$  the kinetic energy (in all what follows the nonrelativistic form  $m v^2/2$  will be used), the averaged Hamiltonian has the form

$$K = W(\alpha, J, \mu) + e \dot{\phi} + (e/c) \alpha \partial \beta / \partial t \quad (7)$$

where  $W$  is independent of  $\beta$  due to the axial symmetry of the dipole field. For adiabaticity to hold, the electric field must be orthogonal to  $\underline{B}$  at least to the lowest order; if such orthogonality is assumed and no time dependence exists, the electric potential  $\phi$  will be a function of  $(\alpha, \beta)$  only and in general, the part of  $K$  not including  $W$  will depend on  $\alpha$  and  $\beta$  only [Northrop and Teller, 1960, eq. 9].

Two limiting cases are easily calculated. If  $J = 0$  the particle is confined to the equatorial plane and one gets

$$W(\alpha, 0, \mu) = \mu B = \mu |g_1^0| / L^3 = P(\mu, L) \quad (8)$$

This form has been widely used in examining the motion of charged particles in the equatorial plane of model magnetospheres [Chen, 1970; Stern, 1974b].

On the other hand, if  $\mu = 0$  then  $v_{||} = v$ . In that case, if  $S$  denotes the total length of the guiding field line, equation (2) reduces to

$$J = 2 m v S = 2 m v I_0 a L \quad (9)$$

where

$$I_0 = 2 + 3^{-\frac{1}{2}} \ln(2 + 3^{\frac{1}{2}}) \approx 2.76$$

Expressing the kinetic energy gives

$$W(\alpha, J, 0) = (J^2/8 \pi I_0^2 a^2) L^{-2} = Q(J, L) \quad (10)$$

Here an approximation will be derived for  $W$  which can be used over the entire range of parameters. In doing so, account should be taken of the special nature of the dipole field, which constrains the allowed form of  $W$ .

The dipole field has the property that all its field lines have the same configuration - only the scale length of the configuration varies. This reduces the functional form of  $W$  from a dependence on 3 variables to a dependence on only two.

For proof, let  $y$  be introduced as auxiliary variable. Then from (1), (6) and (10)

$$I^2(y) = Q I_0^2 / W \quad (11)$$

$$y^2 = P / W \quad (12)$$

If  $y$  is eliminated between (11) and (12) a relation is obtained of the form

$$P / W = F(Q / I_0^2 W) \quad (13)$$

If  $W$  is extracted from this, it can depend on  $\mu$ ,  $J$  and  $\alpha$  only through the functions  $P$  and  $Q$ . A convenient representation of it, with the correct dimensionality and limits, would be

$$W(P, Q) = P + Q + \sum A_k P^{S_k} Q^{1-S_k} \quad (14)$$

where  $0 < S_k < 1$ , since  $Q$  vanishes when  $W = P$  and vice versa.

It simplifies the calculation if  $Q$  is replaced by

$$X = (Q I_0^2 / P)^{\frac{1}{2}} = I(y)/y \quad (15)$$

This reduces (14) to the form

$$W(p, X) = P ( 1 + X^2/I_0^2 + \sum a_k X^{a_k} ) \quad (16)$$

Comparison with (12) shows that the expression in parentheses simply equals  $y^{-2}$ . The results derived here will be presented in this notation.

#### THE APPROXIMATION TO W

One term in the expansion, the term linear in  $X$ , can be derived from the harmonic-oscillator approximation developed by Schulz [1971] for values of  $y$  near unity

$$I(y) \approx 0.7405 ( 1 - y^2 ) \quad (17)$$

If  $y = 1 - \delta$ , then

$$X/0.7405 \approx 2\delta$$

Substituting in (12) gives the coefficient of  $X$  near  $y = 1$  as  $1/0.7405$  (the term proportional to  $X^2$  may be neglected there)

$$\begin{aligned} W &= P / y^2 = P ( 1 + 2\delta ) \\ &= P ( 1 + 1.3505 X ) \end{aligned} \quad (18)$$



A relatively accurate approximation to  $W$ , including two terms in addition to the three developed here, is

$$\begin{aligned}
 W &= P \left( 1 + 1.3505 X - 0.030425 X^{4/3} + 0.10066 X^{5/3} + X^2 / I_0^2 \right) \\
 &= P f(X)
 \end{aligned} \tag{19}$$

As a test we note that by (12) this should satisfy

$$y^2 f(X) = 1 \tag{20}$$

Actual values of this product, as functions of the mirror colatitude  $\theta_m$ , are listed in Table 1. The largest errors fall in the range which in most practical cases is inside the loss cone and for most of the field line the error is 1% or less.

#### APPLICATION

The relation (20) establishes a certain relation between  $X$  and  $y$ , or equivalently between  $X$  and the equatorial pitch angle  $\lambda$  ( $y = \sin \lambda$ ), and this relation is plotted in Figure 1. This plot can be used to determine the energization of particles drifting earthward in a combined electric and magnetic field, in the following way.

Suppose a particle starts at  $L = L_1$  with an equatorial pitch angle  $\lambda_1$  and arrives at  $L = L_2$ : we wish to know by how much has its kinetic energy changed. The initial kinetic energy is readily derived from the given data, for by (12)

$$W = \mu |g_1^0| / L^3 \sin^2 \lambda \tag{21}$$

From Figure 1 the initial value  $X_1$  corresponding to  $\lambda_1$  is read off. By (15), (8) and (10)

$$X = (J^2/a^2 8\mu |g_1^0|)^{\frac{1}{2}} L^{\frac{1}{2}} = \text{constant} \cdot L^{\frac{1}{2}} \quad (22)$$

Therefore the final value of  $X$  is

$$X_2 = X_1 (L_2/L_1)^{\frac{1}{2}} \quad (23)$$

With  $X_2$  having been derived, the corresponding value of the pitch angle  $\lambda_2$  is readily read off Figure 1, after which (21) is used to obtain the final energy  $W_2$ .

From (8) and (10) it is evident that in the limiting cases  $J = 0$  and  $\mu = 0$   $W$  increases as  $L^{-3}$  and  $L^{-2}$ , respectively, and near these limits the above more elaborate method need not be used. It becomes useful in the intermediate range where neither of the adiabatic invariants can be considered to be small.

#### MEAN DRIFT

By the formula developed by Northrop and Teller [1960] the average motion of the particle satisfies

$$\langle \dot{\alpha} \rangle = -(c/e) \partial K / \partial \beta \quad (24)$$

$$\langle \dot{\beta} \rangle = (c/e) \partial K / \partial \alpha \quad (25)$$

The differentiation of the electromagnetic terms in equation (7) is straightforward and therefore only that of  $W$  will be discussed. In the special case of the dipole magnetic field no dependence on  $\beta$  exists, so there is no contribution to  $\langle \dot{\alpha} \rangle$  from this term. For deriving (25) one must substitute

$$X = J (8 \pi m \mu)^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \quad (26)$$

$$P = \mu \alpha^3 / a^3 (g_1^0)^2$$

after which differentiation is straightforward. In order to give the motion an intuitive representation it is useful to map it into the equatorial plane, where every point corresponds to a pair of values of  $(\alpha, \beta)$  and can therefore represent the field line passing through it. Let  $\underline{V}$  be the velocity of the point at which the field line to which the particle is attached crosses that plane. Then if  $\partial/\partial t = 0$  and all variables are evaluated in the equatorial plane

$$\begin{aligned} \langle \dot{\alpha} \rangle &= \underline{V} \cdot \nabla \alpha \\ \langle \dot{\beta} \rangle &= \underline{V} \cdot \nabla \beta \end{aligned} \quad (27)$$

Substituting (27) in (24) and (25) and subtracting the former from the latter then gives

$$\nabla K = (c/e) \underline{V} \times \underline{B} \quad (28)$$

Since  $\underline{V}$  is in the equatorial plane, it has no component parallel to  $\underline{B}$ , from which follows (compare eq. 31, Northrop and Teller, 1960)

$$\underline{V} = (e/cB^2) \underline{B} \times \nabla K \quad (29)$$

This resembles the formula for the magnetic drift velocity in the presence of a force field  $-\nabla K$ , but is actually not the same thing (except in the limit  $J = 0$ , when the particles are confined to the equatorial plane) since  $\underline{V}$  is an averaged drift velocity, averaged over the entire field line and then projected onto the equatorial plane.

Equation (29) provides a method for calculating  $\underline{V}$ ; for instance, in the case of the dipole field or in other axisymmetric fields where  $\beta = a\psi$  one obtains, by resolving  $\underline{B}$  into  $\nabla\alpha \times \nabla\beta$ , the relation

$$\underline{V} = \frac{-3c\mu}{eaL} (1 + 1.1254 X - 0.023664 X^{4/3} + 0.0727 X^{5/3} + 2 X^{2/3} I_0^2) \quad (30)$$

which also follows from (25). This can be evaluated by means of (26) and provides a quick derivation of the averaged magnetic drift velocity (electric drifts can be simply added to it) which in general requires a lengthy calculation.

If the values of  $(J, \mu)$  are set,  $K$  is a function of  $(\alpha, \beta)$  and can be chosen to serve as an Euler potential. Let then  $u(\alpha, \beta)$  be the conjugate Euler potential, i.e. let it satisfy

$$\underline{B} = \nabla K \times \nabla u(\alpha, \beta) \quad (31)$$

The mean rate at which  $u$  changes can be evaluated in the equatorial plane and is, in the absence of explicit time dependence

$$\langle \dot{u} \rangle = \underline{V} \cdot \nabla u = e/c = \text{constant} \quad (32)$$

Taken together,  $K$  and  $u$  describe the mean motion of particles, mapped into the equatorial plane, in a simple intuitive way. By (29)

$$\underline{V} \cdot \nabla K = 0 \quad (33)$$

i.e. the particle stays on a constant value of  $K$  (this no longer holds if  $K$  contains direct dependence on  $t$ ), so that lines of constant  $K$  in the equatorial plane resemble rays along which particles are guided. On the other hand, by (32) lines of constant  $u$  play the role of "wavefronts", since a group of particles starting initially from a given value of  $u$  will at all times share the same value of  $u$ , in the particular representation of  $u$  which was initially chosen (the choice of  $u$  is not unique,

and a group of particles lined up along a line of constant  $u$  in one representation might no longer do so when another one is used).

These "pacing functions" (they pace the rate at which particles advance along lines of constant  $K$ ) will be discussed in a separate paper. The "conjugate potential" of a given electric potential  $\phi(\alpha, \beta)$  [Stern, 1974a] represent the special case of  $u$  when  $W = 0$  and  $\partial/\partial t = 0$  in equation (7).

According to Northrop and Teller, the "bounce time"  $T$  for one complete oscillation between mirror points satisfies

$$\begin{aligned} T^{-1} &= \partial K / \partial J \\ &= (\partial K / \partial X) (\partial X / \partial J) \end{aligned} \quad (34)$$

By (15)  $X$  depends on  $J$  through  $I(y)$  (the denominator there is related to the first invariant  $\mu$ ) and explicitly one then finds

$$X = J / 2 m v a I_y \quad (35)$$

(this may be compared to eq. 9, where  $I$  has attained its maximal value  $I_0$ ). Thus

$$\begin{aligned} T^{-1} &= (\mu |g_1^0| / 8 m a^2 L^5)^{\frac{1}{2}} (1.3505 - 0.040567 X^{1/3} + \\ &+ 0.16777 X^{2/3} + 2 X / I_0^2) \end{aligned} \quad (36)$$

Note that the averaged equatorial drift velocity  $\underline{V}$  and bounce period  $T$  given by equations (30) and (36) respectively reduce to the expressions derived by Schulz [1971] in the limits of  $J = 0$  and  $\mu = 0$ .

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## CAPTIONS TO FIGURE AND TABLE

Figure 1 - A plot of  $X$  (defined in equation 15) against the equatorial pitch angle  $\lambda$  .

Table 1 - Values of the product  $y^2 f(X)$  (given in equation 20) using the approximation (19) for  $f(X)$ , as a function of mirror colatitude  $\theta_m$  and of  $y^2$  .

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Table 1

$\theta_m$ degrees	$\lambda$ degrees	$y^2$	$y^2 \hat{r}(X)$
5	0.03		1.0290
10	0.21		1.0356
15	0.71	0.0002	1.0297
20	1.66	0.0008	1.0179
25	3.17	0.0031	1.0063
30	5.34	0.0087	0.9989
35	8.23	0.0205	0.9966
40	11.89	0.0424	0.9986
45	16.33	0.0791	1.0032
50	21.56	0.1350	1.0075
55	27.58	0.2143	1.0099
60	34.38	0.3189	1.0102
65	41.97	0.4473	1.0081
70	50.32	0.5924	1.0049
74	57.54	0.7120	1.0026
78	65.20	0.8242	1.0010
82	73.23	0.9167	1.0001
86	81.54	0.9782	0.9998



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**X VS. EQUATORIAL PITCH ANGLE  $\lambda$**

